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A NOTE ON ROUGH IDEAL CONVERGENCE OF DOUBLE SEQUENCES

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Abstract: In this article, we invent the notion of rough ideal convergence of double sequence spaces in neutrosophic normed spaces. We investigate some of its topological and algebraic properties of the newly defined concept in neutrosophic normed spaces. We reveal some characterization theorems of it in neutrosophic normed spaces. We also establish its relationships with other known sequence spaces.

Keywords and Phrases: Ideal convergence, rough convergence, double sequence space, neutrosophic normed space.

2020 Mathematics Subject Classification: 03E72, 54A05, 54A40, 54J05.

1. Introduction

Classical methods often fail to solve real-life problems due to uncertainties. To address such situations, Zadeh [21] introduced the notion of fuzzy set theory, which incorporates only membership values. Thereafter, Atanassov [1] extended this concept by developing intuitionistic fuzzy set theory, which uses both membership and non-membership values.

However, decision-making problems under uncertainty were still not fully resolved. Smarandache [17] introduced the notion of neutrosophic sets, where each element is characterized by three independent functions: membership function (T), non-membership function (F), and indeterminacy function (I), defined on the universe of discourse. Further investigations on the applications of this concept were carried out by Smarandache [18].

The concept of intuitionistic fuzzy normed spaces was developed by Saadati [14]. Kastyrko et al. [7] introduced the notion of I-convergence as a generalization of ordinary and statistical convergence. Phu [12, 13] studied rough convergence of sequences in finite-dimensional normed linear spaces and explored their topological and geometric properties.

The concept of rough statistical convergence was further extended to rough ideal convergence by Pal et al. [11]. Dundar [3] investigated rough I_2-convergence of double sequences, while Hossain and Banerjee [6] explored rough I-convergence in intuitionistic fuzzy normed spaces. Kirişci and Şimşek introduced neutrosophic metric spaces [9] and neutrosophic normed spaces (NNS) [8]. Recently, Mursaleen et al. [10] studied the ideal convergence of double sequences in intuitionistic fuzzy normed spaces. Tripathy and Tripathy [20] introduced the norm of I-convergent sequences. Das et al. [2] examined both I and I^*-convergence of double sequences. Subramanian and Esi [19] investigated rough variables of convergence, while Esi and Subramanian [4], and Esi et al. [5] studied triple sequence spaces.

These foundational works inspire us to introduce and study the rough ideal convergence of double sequence spaces in NNS. We investigate their fundamental properties and relationships with other types of convergence in sequence spaces.

The paper is organized as follows. Section 2 briefly recalls some essential definitions and results relevant to our investigation. In Section 3, we introduce and study the notion of rough ideal convergence of double sequences in a NNS. Section 4 concludes the article. Throughout the paper, we consider I as a non-trivial ideal in $\mathbb{N} \times \mathbb{N}$.

2. Preliminaries

In this section, some known results and definitions would be procured for ready reference.

Definition 2.1. [15] *Let X be an universal set. A neutrosophic set A in X is a set contains triplet having truthness, falseness and indeterminacy membership values that can be characterized independently, denoted by T_A, F_A, I_A in $[0,1]$. The neutrosophic set is denoted as follows:*
$A = \{(x, T_A(x), F_A(x), I_A(x)) : x \in X \text{ and } T_A(x), F_A(x), I_A(x) \in [0,1]\}$ *with the*

condition

$$0 \leq T_A(x) + F_A(x) + I_A(x) \leq 3$$

Definition 2.2. [16] *A binary operation* $o : [0,1] \times [0,1] \to [0,1]$ *is called a continuous* t*-norm (TN) if it satisfies the following conditions:*
(i) o *is associative and commutative,*
(ii) o *is continuous,*
(iii) $ao1 = a \forall a \in [0,1]$,
(iv) $a\ o\ c \leq b\ od$ *whenever* $a \leq b$ *and* $c \leq d$ *for each* $a, b, c, d \in [0,1]$.
For example, $aob = a.b$ is a continuous t-norm.

Definition 2.3. [16] *A binary operation* $\bullet : [0,1] \times [0,1] \to [0,1]$ *is called a continuous* t*-conorm (TC) if it satisfies the following conditions:*
(i) $\bullet$ *is associative and commutative,*
(ii) $\bullet$ *is continuous,*
(iii) $a \bullet 1 = a \forall a \in [0,1]$,
(iv) $a \bullet c \leq b \bullet d$ *whenever* $a \leq b$ *and* $c \leq d$ *for each* $a, b, c, d \in [0,1]$.
For example, $a \bullet b = \min\{a + b, 1\}$ is a continuous t-conorm.

Definition 2.4. [8] *Take* F *as a vector space,* $N = \{< u, G(u), B(u), Y(u) >: u \in F\}$ *be a normed space such that* $N : F \times \mathbb{R}^+ \to [0,1]$. *Let* o *and* $\bullet$ *show the continuous TN and continuous TC, respectively. If the following conditions are hold, then the four-tuple* $V = (F, N, 0, 0)$ *is called NNS. For all* $u, v, \in F$ *and* $\lambda, \mu > 0$ *and for each* $\sigma \neq 0$.
(i) $0 \leq G(\mu, \lambda) \leq 1, 0 \leq \beta(\mu, \lambda) \leq 1, 0 \leq Y(\mu, \lambda) \leq 1, \forall \lambda \in \mathbb{R}^+$,
(ii) $G(\mu, \lambda) + \beta(\mu, \lambda) + Y(\mu, \lambda) \leq 3, (\text{ for } \lambda \in \mathbb{R}^+)$,
(iii) $G(\mu, \lambda) = 1(\text{ for } \lambda > 0)$ *iff* $u = 0$,
(iv) $G(u, v, \lambda) = G(v, u, \lambda)(\text{ for } \lambda > 0)$,
(v) $G(u, v, \lambda)\ o\ G(v, u, \lambda) \leq G(u, y, \lambda + u(\forall \lambda, \mu > 0)$,
(vi) $G(u, v,) : .[0, \infty) \to [0,1]$ *is continuous*
(vii) $\lim_{\lambda \to \infty} G(u, v, \lambda) = 1(\forall \lambda > 0)$
(viii) $B(u, v, \lambda) = 0(\text{ for } \lambda > 0)$ *iff* $u = v$,
(ix) $B(u, v, \lambda) = B(v, u, \lambda)(\text{ for } \lambda > 0)$,
(x) $B(u, v, \lambda) \bullet B(v, y, \mu) \geq B(u, y, \lambda + \mu)(\forall \lambda, \mu > 0)$,
(xi) $B(u, v,) : .[0, \infty) \to [0,1]$ *is continuous,*
(xii) $\log_{\lambda \to \infty} B(u, v, \lambda) = 0(\forall \lambda > 0)$,
(xiii) $Y(u, v, \lambda) = 0(\text{ for } \lambda > 0)$ *iff* $u = v$,
(xiv) $Y(u, v, \lambda) = Y(v, u, \lambda)(\forall \lambda > 0)$,
(xv) $Y(u, v, \lambda) \bullet Y(v, y, \mu) \geq Y(u, y, \lambda + \mu)(\forall \lambda, \mu > 0)$,
(xvi) $B(u, v,) : .[0, \infty) \to [0,1]$ *is continuous,*

(xvii) $\lim_{\lambda\to\infty} Y(u,v,\lambda) = 1 (\forall\lambda > 0)$
(xviii) If $\lambda \leq 0$, *then* $G(u,v,\lambda) = 0, B(u,v,\lambda) = 1$ *and* $Y(u,v,\lambda) = 1$.
Then $\eta = (G,B,Y)$ *is called neutrosophic norm (in short, NN).*

Definition 2.5. [7] *If X is a non-empty set, then a family of set* $I \subset P(X)$ *is called an ideal in X if and only if*
(i) For each $A, B \in I$, *we have* $A \cup B \in I$.
(ii) For each $A \in I$ *and* $B \subset A$, *we have* $B \in I$,
where $P(X)$ *is the power set of X.*
I is called non - trivial ideal if $X \notin I$.

Definition 2.6. [7] *Let X be a non - empty set. A non - empty family of sets* $\mathscr{F} \subset P(X)$ *is called a filter on X if and only if*
(i) $\theta \notin \mathscr{F}$.
(ii) For each $A, B \in F$ *we have* $A \cap B \in \mathscr{F}$.
(iii) For each $A \in \mathscr{F}$ *and* $A \subset B$, *we have* $B \in \mathscr{F}$.

Definition 2.7. [7] *A non - trivial ideal I in X is called an admissible ideal if it is different from* $P(\mathbb{N})$ *and it contains all singletons, i.e.,* $\{x\} \in I$ *for each* $x \in X$.

Let $I \subset P(X)$ *be a non - trivial ideal. Then a class* $F(I) = \{M \subset X : M = X \backslash A$, *for some* $A \in I\}$ *is a filter on X, called the filter associated with the ideal I.*

Definition 2.8. [7] *An admissible ideal* $I \subset P(\mathbb{N})$ *is said to satisfy the condition (AP) if for every sequence* $(A_n)_{n\in\mathbb{N}}$ *of pairwise disjoint sets from I, there are sets* $B_n \subset \mathbb{N}, n \in \mathbb{N}$ *such that the symmetric difference* $A_n \Delta B_n$ *is a finite set for every n and* $\cup_{n\in\mathbb{N}} B_n \in I$.

Definition 2.9. [7] *Let* $I \subset 2^{\mathbb{N}}$ *be a non - trivial ideal in* $\mathbb{N}$. *Then a sequence* $x = (x_k)$ *is said to be I - convergent to L if, for every* $\varepsilon > 0$, *the set*

$$\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} \in I$$

In this case, we write $I - \lim x = L$.

Definition 2.10. [7] *Let* $I \subset 2^{\mathbb{N}}$ *be an admissible ideal in* $\mathbb{N}$. *A sequence* $x = (x_k)$ *is called I - Cauchy if, for every* $\varepsilon > 0, \exists$ *a number* $N_0 = N_0(\varepsilon)$ *such that*

$$\{k \in \mathbb{N} : |x_k - x_{N_0}| \geq \varepsilon\} \in I.$$

Definition 2.11. [2] *A double sequence* $\{x_{\mathrm{mn}}\}$ *of real numbers is said to be convergent to* $\xi \in \mathbb{R}$ *if for any* $\varepsilon > 0 \exists\ \mathbb{N}_\varepsilon \in \mathbb{N}$ *such that* $|x_{\mathrm{mn}} - \xi| < \varepsilon \forall m, n \geq \mathbb{N}_\varepsilon$.

Definition 2.12. [2] *A double sequence* $\{x_{mn}\}_{m,n\in\mathbb{N}}$ *of real numbers is said to be* I_2 *- convergent to* $\xi \in \mathbb{R}$ *if for every* $\varepsilon > 0$, *the set* $\{(\mathrm{m},\mathrm{n}) \in \mathrm{N} \times \mathrm{N} : |x_{\mathrm{mn}} - \xi| \geq \varepsilon\} \in I_2$.

3. Main Results

Here, we investigate the concept of rough ideal convergence of double sequences in a NNS.

Definition 3.1. *Let $(F, N, o, \bullet)$ be a NNS with NN $\eta = (G, B, Y)$. For $r > 0$, the open ball $B(x, \lambda, r)$ with center $x \in F$ and radius $0 < \lambda < 1$, is the set* $B(x, \lambda, r) = \{y \in F : G(x - y, r) > 1 - \lambda, B(x - y, r) < \lambda, Y(x - y, r) < \lambda\}$. *Similarly, closed ball is the set* $\overline{B(x, \lambda, r)} = \{y \in F : G(x - y, r) \geq 1 - \lambda, B(x - y, r) \leq \lambda, Y(x - y, r) \leq \lambda\}$.

Definition 3.2. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a NNS $(F, N, o, \bullet)$ with NN $\eta = (G, B, Y)$. Then a point $\gamma \in F$ is called a I - cluster point of $(x_n)_{n\in\mathbb{N}}$ with respect to the NN $\eta = (G, B, Y)$ if for every $\varepsilon > 0, \lambda \in (0, 1)$, the set* $\{n \in \mathbb{N} : G (x_n - \gamma, \varepsilon) > 1 - \lambda, B (x_n - \gamma, \varepsilon) < \lambda)$ *and* $Y (x_n - \gamma, \varepsilon) < \lambda\} \notin I$.

Definition 3.3. *Let $\{x_{mn}\}$ be a double sequence in a NNS $(F, N, o, \bullet)$ with NN $\eta = (G, B, Y)$. Then $\{x_{mn}\}$ is said to be convergent to $\xi \in F$ with respect to the* NN$\eta = (G, B, Y)$ *if for every $\varepsilon > 0$ and $\lambda \in (0, 1), \exists\ \mathbb{N}_\epsilon \in \mathbb{N}$ such that $G (x_{mn} - \xi, \varepsilon) > 1 - \lambda, B (x_{mn} - \xi, \varepsilon) < \lambda$ and $Y (x_{mn} - \xi, \varepsilon) < \lambda \forall m, n \geq \mathbb{N}_\epsilon$. In this case, we write $(G, B, Y) - \lim x_{mn} = \xi$ or $x_{mn} \xrightarrow{(G,B,Y)} \xi$.*

Definition 3.4. *Let I_2 be a non - trivial ideal of $\mathbb{N} \times \mathbb{N}$ and $(F, N, o, \bullet)$ be a NNS with NN $\eta = (G, B, Y)$. A double sequence $x = \{x_{mn}\}$ of elements of F is said to be I_2 - convergent to $L \in F$ if for each $\varepsilon > 0$ and $t > 0$, $\{(m, n) \in \mathbb{N} \times \mathbb{N} : G (x_{mn} - L, t) \leq 1 - \varepsilon$ or $B (x_{mn} - L, t) \geq \varepsilon$ or $Y (x_{mn} - L, t) \geq \varepsilon\} \in I_2$. In this case, we write $I_2^{(G,B,Y)} - \lim x = L$ or $x_{mn} \xrightarrow{I_2^{(G,B,Y)}} L$.*

Definition 3.5. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a NNS $(F, N, o, \bullet)$ with NN $\eta = (G, B, Y)$ and r be a non negative number. Then $(x_n)_{n\in\mathbb{N}}$ is said to be rough I - convergent to $\xi \in F$ with respect to the NN $\eta = (G, B, Y)$ if for every $\varepsilon > 0$ and $\lambda \in (0, 1), \{n \in \mathbb{N} : G (x_n - \xi, r + \varepsilon) \leq 1 - \lambda$ or $B (x_n - \xi, r + \varepsilon) \geq \lambda$ or $Y (x_n - \xi, r + \varepsilon) \geq \lambda\} \in I$. In this case, ξ is called $r - I_{(G,B,Y)}-$ limit of $(x_n)_{n\in\mathbb{N}}$ and we write $r - I_{(G,B,Y)} - \text{limit}\, x_n = \xi$ or $x_n \xrightarrow{r-I_{(G,B,Y)}} \xi$.*

Definition 3.6. *Let $\{x_{mn}\}$ be a double sequence in a NNS $(F, N, o, \bullet)$ with NN $\eta = (G, B, Y)$ and r be a non negative real number. Then $\{x_{mn}\}$ is said to be rough convergent (in short, r - convergent) to $\xi \in X$ with respect to the NN (G, B, Y) if for every $\varepsilon > 0$ and $\lambda \in (0, 1), \exists \mathbb{N}_\lambda \in \mathbb{N}$ such that $G (x_{mn} - \xi, r + \varepsilon) > 1 - \lambda$, $B (x_{mn} - \xi, r + \varepsilon) < \lambda$ and $Y (x_{mn} - \xi, r + \varepsilon) < \lambda \quad \forall m, n \geq \mathbb{N}_\lambda$. In this case, we write $r_2^{(G,B,Y)} \lim x_{mn} = \xi$ or $x_{mn} \xrightarrow{r_2^{(G,B,Y)}} \xi$.*

Definition 3.7. *Let $\{x_{mn}\}$ be a double sequence in a NNS $(F, N, o, \bullet)$ with NN $\eta = (G, B, Y)$ and r be a non negative real number. Then $\{x_{mn}\}$ is said to be rough I_2 - convergent to $\xi \in F$ with respect to the NN (G, B, Y) if for every $\varepsilon > 0$ and $\lambda \in (0, 1), \{(m, n) \in \mathbb{N} \times \mathbb{N} : G(x_{mn} - \xi, r + \varepsilon) \leq 1 - \lambda$ or $B(x_{mn} - \xi, r + \varepsilon) \geq \lambda$ or $Y(x_{mn} - \xi, r + \varepsilon) \geq \lambda\} \in I_2$. In this case, ξ is called $r - I_2^{(G,B,Y)}$ - limit of $\{x_{mn}\}$ and we write $x_{mn} \xrightarrow{r - I_2^{(G,B,Y)}} \xi$.*

Remark 3.1. *(a) If we put $r = 0$ in Definition 3.7, then the notion of rough I_2 - convergence with respect to the NN (G, B, Y) coincides with the notion of I_2 - convergence with respect to the NN (G, B, Y). So, our main interest is on the fact $r > 0$.*
(b) If we use $I_2 = I_2^0$ in Definition 3.7, then the notion of rough I_2 - convergence with respect to the NN(G, B, Y) *coincides with the notion of rough convergence of double sequences with respect to the* NN(G, B, Y).

Note 1. From Definition 3.7, we get $r - I_2^{(G,B,Y)}$ - limit of $\{x_{mn}\}$ is not unique. So, in this regard, we denote $I_2^{(G,B,Y)} - \mathrm{LI}M^r_{x_{mn}}$ to mean the set of all $r - I_2^{(G,B,Y)} -$ limit of $\{x_{mn}\}$, i.e., $I_2^{(G,B,Y)} - \mathrm{LI}M^r_{x_{mn}} = \left\{\xi \in F : x_{mn} \xrightarrow{r - I_2^{(G,B,Y)}} \xi\right\}$. The double sequence $\{\mathrm{x_{mn}}\}$ is called rough I_2 - convergent if $I_2^{(G,B,Y)} - \mathrm{LI}\, M^r_{x_{mn}} \neq \emptyset$.
We denote $\mathrm{LI}\, M^{r(G,B,Y)}_{x_{mn}}$ to mean the set of all rough convergent limits of the double sequence $\{\mathrm{x_{mn}}\}$ with respect to the NN (G, B, Y). The sequence $\{\mathrm{x_{mn}}\}$ is called rough convergent if $\mathrm{LIM}^{r(G,B,Y)}_{x_{mn}} \neq \emptyset$. If the sequence is unbounded, then $\mathrm{LI}\, M^{r(G,B,Y)}_{x_{mn}} = \emptyset$, although in such cases $I_2^{(G,B,Y)} - \mathrm{LI}\, M^r_{x_{mn}} \neq \emptyset$ may happen which will be shown in the following example.

Example 3.1. Let $(F, \|\|.)$ be a real normed linear space and $G(x, t) = \frac{t}{t + \|x\|}$, $B(x, t) = \frac{\|x\|}{t + \|x\|}$ and $Y(x, t) = \frac{\|x\|}{t + \|x\|}$ for all $x \in F$ and $t > 0$. Also, let ao b = ab and a $\bullet$ b = min{a + b, 1}. Then $(F, N, o, \bullet)$ is a NNS with NN $\eta = (G, B, Y)$. Now let us consider ideal I_2 consisting of all those subsets of $\mathbb{N} \times \mathbb{N}$ whose double natural density are zero. Let us consider the double sequence $\{x_{\mathrm{mn}}\}$ by

$$x_{mn} = \begin{cases} (-1)^{m+n}, \text{ if } m, n \neq i^2, i \in \mathbb{N} \\ mn, \text{ otherwise} \end{cases}$$

Then $I_2^{(G,B,Y)} - \mathrm{LIM}^r_{x_{mn}} = \begin{cases} \emptyset, r < 1 \\ [1 - r, r - 1], r \geq 1. \end{cases}$
$\mathrm{LI}\, M^{r(G,B,Y)}_{x_{mn}} = \emptyset$, for any r.

Remark 3.2. *Remark 3.11. From Example 3.10, we have* $I_2^{(G,B,Y)} - \mathrm{LI}\, M^r_{x_{mn}} \neq \emptyset$

does not imply that $\mathrm{LI}\, M_{x_{mn}}^{r(G,B,Y)} \neq \emptyset$. *But, whenever* I_2 *is an admissible ideal then* $\mathrm{LI}\, M_{x_{mn}}^{r(G,B,Y)} \neq \emptyset$ *implies* $I_2^{(G,B,Y)} - \mathrm{LI}\, M_{x_{mn}}^{r} \neq \emptyset$ *as* $I_2^0 \subset I_2$.

Definition 3.8. *Let* $\{x_{mn}\}$ *be a double sequence in a NNS* ($F, N, o, \bullet$) *with NN* (G, B, Y). *Then* $\{x_{mn}\}$ *is said to be* I_2 *- bounded with respect to the* NN(G, B, Y) *if for every* $\lambda \in (0,1), \exists$ *a positive real number* M *such that* $\{(m,n) \in \mathbb{N} \times \mathbb{N} : G(x_{mn}, M) \leq 1 - \lambda$ *or* $B(x_{mn}, M) \geq \lambda$ *or* $Y(x_{mn}, M) \geq \lambda\} \in I_2$.

Theorem 3.1. *Let* $\{x_{mn}\}$ *be a double sequence in a NNS* ($F, N, o, \bullet$) *with respect to the NN* (G, B, Y). *Then* $\{x_{mn}\}$ *is said to be* I_2 *- bounded if and only if* $I_2^{(G,B,Y)} - \mathrm{LI}M_{x_{mn}}^{r} \neq \emptyset$ *for all* $r > 0$.

Proof. First suppose that $\{x_{mn}\}$ is I_2 - bounded with respect to the NN(G, B, Y). Then for every $\lambda \in (0,1) \exists$ a positive real number M such that $\{(m,n) \in \mathbb{N} \times \mathbb{N} : G(x_{mn}, M) \leq 1 - \lambda$ or $B(x_{mn}, M) \geq \lambda$ or $Y(x_{mn}, M) \geq \lambda\} \in I_2$. Let $K = \{(m,n) \in \mathbb{N} \times \mathbb{N} : G(x_{mn}, M) \leq 1 - \lambda$ or $B(x_{mn}, M) \geq \lambda$ or $Y(x_{mn}, M) \geq \lambda\}$. Now for $(i,j) \in K^c$, we have $G(x_{ij} - \theta, r + M) \geq G(x_{ij}, M)\, oG(\theta, r) > (1-\lambda)o1 = 1-\lambda, B(x_{ij} - \theta, r + M) \leq B(x_{ij}, M) \bullet B(\theta, r) < \lambda \bullet 0 = \lambda$ and $B(x_{ij} - \theta, r + M) \leq B(x_{ij}, M) \bullet B(\theta, r) < \lambda \bullet 0 = \lambda$, where θ is the zero element of F. Therefore $\{(i,j) \in \mathbb{N} \times \mathbb{N} : G(x_{ij} - \theta, r + M) \leq 1 - \lambda$ or $B(x_{mn}, M) \geq \lambda$ or $Y(x_{mn}, M) \geq \lambda\} \subset K$. Since $K \in I_2, \theta \in I_2$. Hence $I_2^{(G,B,Y)} - \mathrm{LI}M_{x_{mn}}^{r} \neq \emptyset$.

Conversely, suppose that $I_2^{(G,B,Y)} - \mathrm{LI}_{x_{mn}}^{r} \neq \emptyset$. Then $\exists \beta \in I_2^{(G,B,Y)} - \mathrm{LI}M_{x_{mn}}^{r}$ such that for every $\varepsilon > 0$ and $\lambda \in (0,1)$ such that $\{(m,n) \in \mathbb{N} \times \mathbb{N} : G(x_{mn} - \beta, r + \varepsilon) \leq 1 - \lambda.$ or $B(x_{mn} - \beta, r + \varepsilon) \geq \lambda$ or $Y(x_{mn} - \beta, r + \varepsilon) \geq \lambda\} \in I_2$. This shows that almost all x_{mn} are contained in some ball with center β. Hence $\{x_{mn}\}$ is I_2 - bounded. This completes the proof.

Now we will discuss on some algebraic characterization of rough I_2 - convergence in a NNS.

Theorem 3.2. *Let* $\{x_{mn}\}$ *and* $\{y_{mn}\}$ *be two double sequences in a NNS* ($F, \eta, o, \bullet$). *Here* $\eta = (G, B, Y)$ *is the NN. Then for some* $r > 0$, *the following statements hold:*

(i) If $x_{mn} \xrightarrow{r-I_2^{(G,B,Y)}} \xi$ *and* $y_{mn} \xrightarrow{r-I_2^{(G,B,Y)}} \eta$, *then* $x_{mn} + y_{mn} \xrightarrow{r-I_2^{(G,B,Y)}} \xi + \eta$.

(ii) If $x_{mn} \xrightarrow{r-I_2^{(G,B,Y)}} \xi$ *and* $k(\neq 0) \in \mathbb{R}$, *then* $kx_{mn} \xrightarrow{r-I_2^{(G,B,Y)}} k\xi$.

Proof. Let $\{x_{mn}\}$ and $\{y_{mn}\}$ be two double sequences in a NNS $(F, \eta, o, \bullet)$. Here $\eta = (G, B, Y)$ is the NN. Take $r > 0$ and $\lambda \in (0,1)$.

(i) Let $x_{mn} \xrightarrow{r-I_2^{(G,B,Y)}} \xi$ and $y_{mn} \xrightarrow{r-I_2^{(G,B,Y)}} \eta$. Also, let $\varepsilon > 0$ be given. Now, for a given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s) \circ (1-s) > 1 - \lambda$ and

$s \bullet s < \lambda$. Then $A, B \in I_2$, where

$$A = \{(m,n) \in \mathbb{N} \times \mathbb{N} : G\left(x_{mn} - \xi, \frac{r+\varepsilon}{2}\right) \leq 1 - s \text{ or } B\left(x_{mn} - \xi, \frac{r+\varepsilon}{2}\right) \geq s \text{ or } Y\left(x_{mn} - \xi, \frac{r+\varepsilon}{2}\right) \geq s\}$$

and

$$B = \{(m,n) \in \mathbb{N} \times \mathbb{N} : G\left(y_{mn} - \eta, \frac{r+\varepsilon}{2}\right) \leq 1 - s \text{ or } B\left(y_{mn} - \eta, \frac{r+\varepsilon}{2}\right) \geq s \text{ or } Y\left(y_{mn} - \eta, \frac{r+\varepsilon}{2}\right) \geq s\}$$

So $A^c \cap B^c \in \mathcal{F}(I_2)$. Now for $(i,j) \in A^c \cap B^c$, we have

$$\begin{aligned} G\left(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon\right) &\geq G\left(x_{ij} - \xi, \frac{r+\varepsilon}{2}\right) \circ G\left(y_{ij} - \eta, \frac{r+\varepsilon}{2}\right) \\ &> (1-s) \circ (1-s) \\ &> 1 - \lambda, \\ B\left(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon\right) &\leq B\left(x_{ij} - \xi, \frac{r+\varepsilon}{2}\right) \bullet B\left(y_{ij} - \eta, \frac{r+\varepsilon}{2}\right) \\ &< s \bullet s \\ &< \lambda \end{aligned}$$

and

$$\begin{aligned} Y\left(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon\right) &\leq Y\left(x_{ij} - \xi, \frac{r+\varepsilon}{2}\right) \bullet Y\left(y_{ij} - \eta, \frac{r+\varepsilon}{2}\right) \\ &< s \bullet s \\ &< \lambda. \end{aligned}$$

Therefore

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : G\left(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon\right) \leq 1 - \lambda \text{ or } B\left(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon\right) \geq \lambda \text{ or } Y\left(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon\right) \geq \lambda\} \subset A \cup B.$$

Since $A \cup B \in I_2$,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : G\left(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon\right) \leq 1 - \lambda \\ \text{or } B\left(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon\right) \geq \lambda \\ \text{or } Y\left(x_{ij} + y_{ij} - (\xi + \eta), r + \varepsilon\right) \geq \lambda\} \in I_2.$$

Therefore $x_{mn} + y_{mn} \xrightarrow{r-I_2^{(G,B,Y)}} \xi + \eta$.

(ii) Let $x_{mn} \xrightarrow{r-I_2^{(G,B,Y)}} \xi$ and $k(\neq 0) \in \mathbb{R}$. Then

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : G\left(x_{mn} - \xi, \frac{r+\varepsilon}{|k|}\right) \leq 1 - \lambda \\ \text{or } B\left(x_{mn} - \xi, \frac{r+\varepsilon}{|k|}\right) \geq \lambda \\ \text{or } Y\left(x_{mn} - \xi, \frac{r+\varepsilon}{|k|}\right) \geq \lambda\} \in I_2.$$

Therefore,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : G\left(kx_{mn} - k\xi, r + \varepsilon\right) \leq 1 - \lambda \\ \text{or } B\left(kx_{mn} - k\xi, r + \varepsilon\right) \geq \lambda \\ \text{or } Y\left(kx_{mn} - k\xi, r + \varepsilon\right) \geq \lambda\} \in I_2.$$

Hence $kx_{mn} \xrightarrow{r-I_2^{(G,B,Y)}} k\xi$. This completes the proof.

Now we prove some topological and geometrical properties of the set $I_2^{(G,B,Y)} - \mathrm{LI}\,M^r_{x_{mn}}$.

Theorem 3.3. *Let $\{x_{mn}\}$ be a double sequence in a NNS $(F, N, o, \bullet)$. Here $\eta = (G, B, Y)$ is the NN. Then $\forall\ r > 0$, the set $I_2^{(G,B,Y)} - \mathrm{LIM}^r_{x_{mn}}$ is closed.*

Proof. If $I_2^{(G,B,Y)} - \mathrm{LIM}^r_{x_{mn}} = \emptyset$ then there is nothing to prove. So, let $I_2^{(G,B,Y)} - \mathrm{LIM}^r_{x_{mn}} \neq \emptyset$. Suppose that $\{z_{mn}\}$ is a double sequence in $I_2^{(G,B,Y)} - \mathrm{LIM}^r_{x_{mn}}$ such that $z_{mn} \xrightarrow{(G,B,Y)} y_0$.

Now, for a given $\lambda \in (0,1)$, choose $s \in (0,1)$ such that $(1-s)\circ(1-s) > 1-\lambda$ and $s \bullet s < \lambda$. Let $\varepsilon > 0$ be given. Then $\exists m_0 \in \mathbb{N}$ such that

$$G\left(z_{mn} - y_0, \frac{\varepsilon}{2}\right) > 1-s,$$
$$B\left(z_{mn} - y_0, \frac{\varepsilon}{2}\right) < s,$$
$$Y\left(z_{mn} - y_0, \frac{\varepsilon}{2}\right) < s$$

$\forall m, n \geq m_0$.

Suppose $i, j > m_0$. Then

$$G\left(z_{ij} - y_0, \frac{\varepsilon}{2}\right) > 1-s,$$
$$B\left(z_{ij} - y_0, \frac{\varepsilon}{2}\right) < s,$$
$$Y\left(z_{ij} - y_0, \frac{\varepsilon}{2}\right) < s.$$

Also,

$$P = \{(m,n) \in \mathbb{N}\times\mathbb{N} : G\left(x_{mn} - z_{ij}, r + \frac{\varepsilon}{2}\right) \leq 1-s \text{ or } B\left(x_{mn} - z_{ij}, r + \frac{\varepsilon}{2}\right) \geq s \text{ or } Y\left(x_{mn} - z_{ij}, r + \frac{\varepsilon}{2}\right) \geq s\} \in I_2.$$

Now, for $(p,q) \in P^c$, we have

$$\begin{aligned} G(x_{pq} - y_0, r+\varepsilon) &\geq G\left(x_{pq} - z_{ij}, r + \frac{\varepsilon}{2}\right) \circ G\left(z_{ij} - y_0, \frac{\varepsilon}{2}\right) \\ &> (1-s)\circ(1-s) \\ &> 1-\lambda, \\ B(x_{pq} - y_0, r+\varepsilon) &\leq B\left(x_{pq} - z_{ij}, r + \frac{\varepsilon}{2}\right) \bullet B\left(z_{ij} - y_0, \frac{\varepsilon}{2}\right) \\ &< s \bullet s \\ &< \lambda, \end{aligned}$$

and

$$\begin{aligned} Y(x_{pq} - y_0, r+\varepsilon) &\leq Y\left(x_{pq} - z_{ij}, r + \frac{\varepsilon}{2}\right) \bullet Y\left(z_{ij} - y_0, \frac{\varepsilon}{2}\right) \\ &< s \bullet s \\ &< \lambda. \end{aligned}$$

Therefore

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : G\left(x_{mn}-y_0, r+\varepsilon\right) \leq 1-\lambda \text{ or } B\left(x_{mn}-y_0, r+\varepsilon\right) \geq \lambda \text{ or } Y\left(x_{mn}-y_0, r+\varepsilon\right) \geq \lambda\} \subset P.$$

Since $P \in I_2$, $y_0 \in I_2^{(G,B,Y)} - \mathrm{LIM}^r_{x_{mn}}$. Therefore $I_2^{(G,B,Y)} - \mathrm{LIM}^r_{x_{mn}}$ is closed. This completes the proof.

Theorem 3.4. *Let $\{x_{mn}\}$ be a double sequence in a NNS $(F, N, o, \bullet)$. Here $\eta = (G, B, Y)$ is the NN. Then $\forall\ r > 0$, the set $I_2^{(G,B,Y)} - \mathrm{LIM}^r_{x_{mn}}$ is convex.*

Proof. Let $x_1, x_2 \in I_2^{(G,B,Y)} - \mathrm{LIM}^r_{x_{mn}}$ and $\kappa \in (0,1)$. Let $\lambda \in (0,1)$. Choose $s \in (0,1)$ such that $(1-s) \circ (1-s) > 1-\lambda$ and $s \bullet s < \lambda$. Then for every $\varepsilon > 0$, the sets $H, T \in I_2$ where

$$H = \{(m,n) \in \mathbb{N}\times\mathbb{N} : G\left(x_{mn}-x_1, \frac{r+\varepsilon}{2(1-\kappa)}\right) \leq 1-s \text{ or } B\left(x_{mn}-x_1, \frac{r+\varepsilon}{2(1-\kappa)}\right) \geq s \text{ or } Y\left(x_{mn}-x_1, \frac{r+\varepsilon}{2(1-\kappa)}\right) \geq s\}$$

and

$$T = \{(m,n) \in \mathbb{N}\times\mathbb{N} : G\left(x_{mn}-x_2, \frac{r+\varepsilon}{2\kappa}\right) \leq 1-s \text{ or } B\left(x_{mn}-x_2, \frac{r+\varepsilon}{2\kappa}\right) \geq s \text{ or } Y\left(x_{mn}-x_2, \frac{r+\varepsilon}{2\kappa}\right) \geq s\}.$$

Now for $(m,n) \in H^c \cap T^c$, we have

$$\begin{aligned} G\left(x_{mn}-[(1-\kappa)x_1+\kappa x_2], r+\varepsilon\right) &\geq G\left((1-\kappa)\left(x_{mn}-x_1\right), \frac{r+\varepsilon}{2}\right) \circ G\left(\kappa\left(x_{mn}-x_2\right), \frac{r+\varepsilon}{2}\right) \\ &= G\left(x_{mn}-x_1, \frac{r+\varepsilon}{2(1-\kappa)}\right) \circ G\left(x_{mn}-x_2, \frac{r+\varepsilon}{2\kappa}\right) \\ &> (1-s) \circ (1-s) \\ &> 1-\lambda, \end{aligned}$$

$$\begin{aligned} B\left(x_{mn}-\left[(1-\kappa)x_1+\kappa x_2\right], r+\varepsilon\right) &\leq B\left((1-\kappa)\left(x_{mn}-x_1\right), \frac{r+\varepsilon}{2}\right) \bullet B\left(\kappa\left(x_{mn}-x_2\right), \frac{r+\varepsilon}{2}\right) \\ &= B\left(x_{mn}-x_1, \frac{r+\varepsilon}{2(1-\kappa)}\right) \bullet B\left(x_{mn}-x_2, \frac{r+\varepsilon}{2\kappa}\right) \\ &< s \bullet s \\ &< \lambda, \end{aligned}$$

and

$$\begin{aligned} Y\left(x_{mn}-\left[(1-\kappa)x_1+\kappa x_2\right], r+\varepsilon\right) &\leq Y\left((1-\kappa)\left(x_{mn}-x_1\right), \frac{r+\varepsilon}{2}\right) \bullet Y\left(\kappa\left(x_{mn}-x_2\right), \frac{r+\varepsilon}{2}\right) \\ &= Y\left(x_{mn}-x_1, \frac{r+\varepsilon}{2(1-\kappa)}\right) \bullet Y\left(x_{mn}-x_2, \frac{r+\varepsilon}{2\kappa}\right) \\ &< s \bullet s \\ &< \lambda, \end{aligned}$$

which gives that

$$\begin{aligned} \{(m,n) \in \mathbb{N}\times\mathbb{N} : G\left(x_{mn}-\left[(1-\kappa)x_1+\kappa x_2\right], r+\varepsilon\right) \leq 1-\lambda \\ \text{or } B\left(x_{mn}-\left[(1-\kappa)x_1+\kappa x_2\right], r+\varepsilon\right) \geq \lambda \\ \text{or } Y\left(x_{mn}-\left[(1-\kappa)x_1+\kappa x_2\right], r+\varepsilon\right) \geq \lambda\} \subset H \cup T. \end{aligned}$$

Since $H \cup T \in I_2$, $(1-\kappa)x_1+\kappa x_2 \in I_2^{(G,B,Y)} - \text{LIM}^r_{x_{mn}}$. Therefore $I_2^{(G,B,Y)} - \text{LIM}^r_{x_{mn}}$ is convex. This completes the proof.

Theorem 3.5. *A double sequence $\{x_{mn}\}$ in a NNS ($F, N, o, \bullet$) is rough I_2 - convergent to $\beta \in F$ with respect to the NN $\eta = (G,B,Y)$ for some $r > 0$ if $\exists$ a double sequence $\{y_{mn}\}$ in F such that $y_{mn} \xrightarrow{I_2^{(G,B,Y)}} \beta$ and for every $\lambda \in (0,1), G\left(x_{mn}-y_{mn}\right) > 1-\lambda, B\left(x_{mn}-y_{mn}\right) < \lambda$ and $Y\left(x_{mn}-y_{mn}\right) < \lambda \forall m,n \in \mathbb{N}$.*

Proof. Let $\varepsilon > 0$ be given. Now, for a $\lambda \in (0,1)$, choose $s \in (0,1)$ such that

$$(1-s) \circ (1-s) > 1-\lambda \quad \text{and} \quad s \bullet s < \lambda.$$

Suppose that $y_{mn} \xrightarrow{I_2^{(G,B,Y)}} \beta$ and

$$G(x_{mn}-y_{mn}) > 1-s, \quad B(x_{mn}-y_{mn}) < s, \quad Y(x_{mn}-y_{mn}) < s \quad \forall\, m,n \in \mathbb{N}.$$

Then the set

$$P = \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \begin{array}{l} G(y_{mn} - \beta, \varepsilon) \leq 1 - s \text{ or} \\ B(y_{mn} - \beta, \varepsilon) \geq s \text{ or} \\ Y(y_{mn} - \beta, \varepsilon) \geq s \end{array} \right\} \in I_2.$$

Now, for $(i,j) \in P^c$, we have:

$$\begin{aligned} G(x_{ij} - \beta, r + \varepsilon) &\geq G(x_{ij} - y_{ij}, r) \circ G(y_{ij} - \beta, \varepsilon) \\ &> (1-s) \circ (1-s) > 1 - \lambda, \end{aligned}$$

$$\begin{aligned} B(x_{ij} - \beta, r + \varepsilon) &\leq B(x_{ij} - y_{ij}, r) \bullet B(y_{ij} - \beta, \varepsilon) \\ &< s \bullet s < \lambda, \end{aligned}$$

$$\begin{aligned} Y(x_{ij} - \beta, r + \varepsilon) &\leq Y(x_{ij} - y_{ij}, r) \bullet Y(y_{ij} - \beta, \varepsilon) \\ &< s \bullet s < \lambda. \end{aligned}$$

Therefore,

$$\begin{aligned} \{(i,j) \in \mathbb{N} \times \mathbb{N} : &G(x_{ij} - \beta, r + \varepsilon) \leq 1 - \lambda \text{ or} \\ &B(x_{ij} - \beta, r + \varepsilon) \geq \lambda \text{ or} \\ &Y(x_{ij} - \beta, r + \varepsilon) \geq \lambda\} \subset P. \end{aligned}$$

Since $P \in I_2$, it follows that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : G(x_{ij} - \beta, r + \varepsilon) \leq 1 - \lambda \text{ or } B(x_{ij} - \beta, r + \varepsilon) \geq \lambda \text{ or } Y(x_{ij} - \beta, r + \varepsilon) \geq \lambda\} \in I_2.$$

Therefore, $\{x_{mn}\}$ is rough I_2-convergent to β with respect to the NN (G, B, Y). This completes the proof.

Definition 3.9. *Let $\{x_{mn}\}$ be a double sequence in a NNS $(F, N, o, \bullet)$. Here $\eta = (G, B, Y\ )$ is the NN. Then a point $\zeta \in F$ is said to be I_2 - cluster point of $\{x_{mn}\}$ with respect to the NN (G, B, Y) if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, $\{(m,n) \in \mathbb{N} \times \mathbb{N} : G\left(x_{mn} - \zeta, \varepsilon\right) > 1 - \lambda, B\left(x_{mn} - \zeta, \varepsilon\right) < \lambda$ and $Y\left(x_{mn} - \zeta, \varepsilon\right) < \lambda\} \notin I_2$. We denote $\Gamma_{(x_{mn})}\left(I_2^{(G,B,Y)}\right)$ to mean the set of all I_2 - cluster points of $\{x_{mn}\}$ with respect to the NN (G, B, Y).*

Definition 3.10. *Let $\{x_{mn}\}$ be a double sequence in a NNS $(F, N, \circ, \bullet)$ and let $r \geq 0$. Here, $\eta = (G, B, Y)$ is the NN. Then a point $\beta \in F$ is said to be a rough I_2-cluster point of $\{x_{mn}\}$ with respect to the NN (G, B, Y) if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, the set*

$$\begin{aligned} \{(m,n) \in \mathbb{N} \times \mathbb{N} : &G(x_{mn} - \beta, r + \varepsilon) > 1 - \lambda, \\ &B(x_{mn} - \beta, r + \varepsilon) < \lambda, \\ &Y(x_{mn} - \beta, r + \varepsilon) < \lambda\} \notin I_2. \end{aligned}$$

We denote $\Gamma^r_{(x_{mn})}\left(I_2^{(G,B,Y)}\right)$ *to mean the set of all rough* I_2 *- cluster points of* $\{x_{mn}\}$ *with respect to the NN* (G, B, Y).

Theorem 3.6. *Let* $\{x_{mn}\}$ *be a double sequence in a NNS* $(F, N, o, \bullet)$. *Here* $\eta = (G, B, Y)$ *is the NN. Then* $\forall\ r > 0$, *the set* $\Gamma^r_{x_{mn}}\left(I_2^{(G,B,Y)}\right)$ *is closed with respect to the NN* (G, B, Y).

Proof. The proof is almost similar to the proof of Theorem 3.3. So we omit details.

Theorem 3.7. *Let* $\{x_{mn}\}$ *be a double sequence in a NNS* $(F, N, o, \bullet)$. *Here* $\eta = (G, B, Y)$ *is the NN. Then for an arbitrary* $x_1 \in \Gamma_{(x_{mn})}\left(I_2^{(G,B,Y)}\right)$ *and* $\lambda \in (0, 1)$, *we have* $G(x_2 - x_1, r) > 1 - \lambda, B(x_2 - x_1, r) < \lambda$ *and* $Y(x_2 - x_1, r) < \lambda$ *for all* $x_2 \in \Gamma^r_{x_{mn}}\left(I_2^{(G,B,Y)}\right)$.

Proof. For a given $\lambda \in (0, 1)$, choose $s \in (0, 1)$ such that

$$(1 - s) \circ (1 - s) > 1 - \lambda \quad \text{and} \quad s \bullet s < \lambda.$$

Since $x_1 \in \Gamma_{(x_{mn})}\left(I_2^{(G,B,Y)}\right)$, for every $\varepsilon > 0$, we have:

$$\Big\{(m,n) \in \mathbb{N} \times \mathbb{N} : G(x_{mn} - x_1, \varepsilon) > 1 - s, \\ B(x_{mn} - x_1, \varepsilon) < s, \\ Y(x_{mn} - x_1, \varepsilon) < s\Big\} \notin I_2.$$

Let (i, j) be in the above set. Then:

$$\begin{aligned} G(x_{ij} - x_2, r + \varepsilon) &\geq G(x_{ij} - x_1, \varepsilon) \circ G(x_1 - x_2, r) \\ &> (1 - s) \circ (1 - s) > 1 - \lambda, \\ B(x_{ij} - x_2, r + \varepsilon) &< s \bullet s < \lambda, \\ Y(x_{ij} - x_2, r + \varepsilon) &< s \bullet s < \lambda. \end{aligned}$$

Thus,

$$\Big\{(m,n) \in \mathbb{N} \times \mathbb{N} : G(x_{mn} - x_2, \varepsilon) > 1 - s, \\ B(x_{mn} - x_2, \varepsilon) < s, \\ Y(x_{mn} - x_2, \varepsilon) < s\Big\} \notin I_2.$$

Hence, $x_2 \in \Gamma^r_{x_{mn}}\left(I_2^{(G,B,Y)}\right)$. This completes the proof.

Theorem 3.8. *Let $\{x_{mn}\}$ be a double sequence in a NNS $(F, N, o, \bullet)$. Here $\eta = (G, B, Y)$ is the NN. Then for some $r > 0, \lambda \in (0, 1)$ and $x_0 \in F$, we have*

$$\Gamma^r_{x_{mn}}\left(I_2^{(G,B,Y)}\right) = \bigcup_{x_0 \in \Gamma_{(x_{mn})}\left(I_2^{(G,B,Y)}\right)} \overline{B\left(x_0, \lambda, r\right)}$$

Proof. For a given $\lambda \in (0, 1)$, choose $s \in (0, 1)$ such that $(1 - s) \circ (1 - s) > 1 - \lambda$ and $s \bullet s < \lambda$.

Let $y_0 \in \bigcup_{x_0 \in \Gamma(x_{mn})(I_2^{(G,B,Y)})} \overline{B(x_0, \lambda, r)}$. Then there exists $x_0 \in \Gamma(x_{mn})(I_2^{(G,B,Y)})$ such that

$$G(x_0 - y_0, r) > 1 - s, \quad B(x_0 - y_0, r) < s, \quad Y(x_0 - y_0, r) < s.$$

Now, since $x_0 \in \Gamma(x_{mn})(I_2^{(G,B,Y)})$, for every $\varepsilon > 0$, there exists a set

$$M = \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \begin{array}{c} G(x_{mn} - x_0, \varepsilon) > 1 - s, \\ B(x_{mn} - x_0, \varepsilon) < s, \\ Y(x_{mn} - x_0, \varepsilon) < s \end{array} \right\} \notin I_2.$$

Let $(i, j) \in M$. Then:

$$\begin{aligned} G(x_{ij} - y_0, r + \varepsilon) &\geq G(x_{ij} - x_0, \varepsilon) \circ G(x_0 - y_0, r) > (1 - s) \circ (1 - s) > 1 - \lambda, \\ B(x_{ij} - y_0, r + \varepsilon) &\leq B(x_{ij} - x_0, \varepsilon) \bullet B(x_0 - y_0, r) < s \bullet s < \lambda, \\ Y(x_{ij} - y_0, r + \varepsilon) &\leq Y(x_{ij} - x_0, \varepsilon) \bullet Y(x_0 - y_0, r) < s \bullet s < \lambda. \end{aligned}$$

Therefore,

$$M \subset \left\{ (i, j) \in \mathbb{N} \times \mathbb{N} : \begin{array}{c} G(x_{ij} - y_0, r + \varepsilon) > 1 - \lambda, \\ B(x_{ij} - y_0, r + \varepsilon) < \lambda, \\ Y(x_{ij} - y_0, r + \varepsilon) < \lambda \end{array} \right\}.$$

Since $M \notin I_2$, the above set is not in I_2. Hence, $y_0 \in \Gamma^r_{x_{mn}}(I_2^{(G,B,Y)})$. Therefore,

$$\bigcup_{x_0 \in \Gamma(x_{mn})(I_2^{(G,B,Y)})} \overline{B(x_0, \lambda, r)} \subseteq \Gamma^r_{x_{mn}}(I_2^{(G,B,Y)}).$$

Conversely, suppose $y_* \in \Gamma^r_{x_{mn}}(I_2^{(G,B,Y)})$. We shall show that

$$y_* \in \bigcup_{x_0 \in \Gamma(x_{mn})(I_2^{(G,B,Y)})} \overline{B(x_0, \lambda, r)}.$$

If possible, suppose

$$y_* \notin \bigcup_{x_0 \in \Gamma(x_{mn})(I_2^{(G,B,Y)})} \overline{B(x_0, \lambda, r)}.$$

Then for every $x_0 \in \Gamma(x_{mn})(I_2^{(G,B,Y)})$,

$$G(x_0 - y_*, r) \leq 1 - \lambda, \quad B(x_0 - y_*, r) \geq \lambda, \quad Y(x_0 - y_*, r) \geq \lambda.$$

Now, by Theorem 3.7,

$$G(x_0 - y_*, r) > 1 - \lambda, \quad B(x_0 - y_*, r) < \lambda, \quad Y(x_0 - y_*, r) < \lambda,$$

which is a contradiction.

Therefore,

$$\Gamma^r_{x_{mn}}(I_2^{(G,B,Y)}) \subseteq \bigcup_{x_0 \in \Gamma(x_{mn})(I_2^{(G,B,Y)})} \overline{B(x_0, \lambda, r)}.$$

Hence,

$$\Gamma^r_{x_{mn}}(I_2^{(G,B,Y)}) = \bigcup_{x_0 \in \Gamma(x_{mn})(I_2^{(G,B,Y)})} \overline{B(x_0, \lambda, r)}.$$

This completes the proof.

4. Conclusion

In this article, we have introduced rough ideal convergence of double sequence spaces in NNS. We have investigated its different algebraic and topological properties, like boundedness, closedness, convexcity etc. in NNS. We have established some characterization theorems in NNS.

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